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AN UPPER BOUND ON THE PROBABILITY DENSITY FUNCTION IN TERMS OF FISHER INFORMATION

A Bound to the Density Function,

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Summary

An inequality for members of a Sobolev space of order one is demonstrated and as a corollary an upper bound, in terms of the Fisher information, is derived for a density function. Also two characterizations of the Laplace and the exponential distribution are indicated.

AN UPPER BOUND ON THE PROBABILITY DENSITY FUNCTION IN TERMS OF FISHER INFORMATION

A BOUND TO THE DENSITY FUNCTION

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1. Introduction. Let $||\cdot||_\infty$, $||\cdot||_2$ denote the $\sup_{\mathbb{R}}$ and $L_2(\mathbb{R})$ norms respectively, and let g' denote the first distributional derivative of the function g in $L_2(\mathbb{R})$. We will demonstrate inequalities

$$(1.1) \qquad ||g\,g'||_2^2 \,/\, ||g'||_2^2 \;\le\; ||g||_\infty^2 \;\le\; ||g||_2\,||g'||_2 \,,$$

and a generalization thereof (see (2.3)), for members g of a Sobolev space of order one. Since $||g||_2$, $||g'||_2 < +\infty$ imply $||gg'||_2 < +\infty$, through the relations $||gg'||_2^2 \le ||g||_\infty^2\,||g'||_2^2 \le ||g||_2\,||g'||_2^3$, and the lower bound on $||g||_\infty^2$ in (1.1) is immediate we need derive only the upper bound. An interesting special case of inequalities (1.1), when $f = g^2$ is a probability density function on the real line with finite Fisher information $I(f) = \int f'^2/f$, is

$$(1.2) \qquad ||f'||_2^2 \,/\, I(f) \;\le\; ||f||_\infty \;\le\; I(f)^{1/2}/2 \,,$$

with equality in the rightmost hand side if and only if f is the density or the Laplace distribution. A generalization of (1.2) is given by (3.1) below.

It should be noted that a weaker form of the upper bound in (1.1) appears in Tapia and Thompson (1978), page 116, for absolutely continuous functions. Their proof can be modified to achieve the upper bound in (1.1), but our proof allows slightly more general functions.

2. An inequality for members of a Sobolev space. Let $\mathbb{R}$ denote the real line and let $L_2 \equiv L_2(\mathbb{R})$ be the space of all Lebesque measurable square integrable functions $g : \mathbb{R} \to \mathbb{R}$. It is well known that $(L_2\,,\,\langle\cdot\,,\,\rangle_2)$ is a Hilbert space for the inner product $\langle g,h\rangle_2 \equiv \int_{\mathbb{R}} gh$

and the induced norm $||g||_2 \equiv (\int_{\mathbb{R}} g^2)^{\frac{1}{2}}$. By a Sobolev space of order one on $\mathbb{R}$, denoted by $H \equiv H^1(\mathbb{R})$, is meant the space of all functions $g : \mathbb{R} \to \mathbb{R}$ such that $g, g' \in L_2$, where g' denotes the distributional derivative of order one of g. Endowed with the inner product $\langle g,h \rangle_{\underline{\alpha}} \equiv \alpha_0 \langle g,h \rangle_2 + \alpha_1 \langle g',h' \rangle_2$, where $\underline{\alpha} = (\alpha_0, \alpha_1)$ with $\alpha_0, \alpha_1 > 0$, and the induced norm $||g||_{\underline{\alpha}} \equiv (\langle g,g \rangle_{\underline{\alpha}})^{\frac{1}{2}}$, H is a reproducing kernel Hilbert space (RKHS), i.e. there exists a unique real valued function $k(x,y)$ on $\mathbb{R} \times \mathbb{R}$, called the reproducing kernel (RK), such that,

$$k_y(\cdot) \equiv k(\cdot,y) \in H \quad \text{for all} \quad y \in \mathbb{R} ,$$

and

$$\langle g,k_y \rangle_{\underline{\alpha}} = g(y) \quad \text{for all} \quad g \in H .$$

The kernel of H is given by

$$(1.1) \qquad k_{\underline{\alpha}}(x,y) = (4\alpha_0\alpha_1)^{-\frac{1}{2}} \exp\{-(\alpha_0/\alpha_1)^{\frac{1}{2}}|x-y|\} .$$

For a treatment of the RKHS and Sobolev spaces the reader is referred to Aronszajn (1950), Parzen (1967), Gel'fand and Shilov (1964) and Yosida (1974).

We next derive an inequality which is valid for all functions in $H^1(\mathbb{R})$. A discrete version of this inequality is now available also, due to Professor J. Keilson (personal communication), namely

$$\max g_n^2 \le (\Sigma g_n^2)^{\frac{1}{2}} (\Sigma (g_n - g_{n-1})^2)^{\frac{1}{2}} ,$$

for all sequences $(g_n)_{-\infty}^{+\infty}$ in ℓ_2.

Proposition 2.1. For every $g \in H$ we have

$$||g||_\infty^2 \le ||g||_2 \, ||g'||_2 . \tag{2.2}$$

Proof: We endow H with the collection of norms $||\cdot||_{\underline{\alpha}}$ induced by the inner products $\langle\cdot,\cdot\rangle_{\underline{\alpha}}$, defined above. For every $\underline{\alpha} = (\alpha_0,\alpha_1)$ with $\alpha_0,\alpha_1 > 0$, $(H, \langle\cdot,\cdot\rangle_{\underline{\alpha}})$ is a RKHS with RK $k_{\underline{\alpha}}$ given by (2.1).

Using the Cauchy-Schwarz inequality we have for every $u \in H$

$$|g(x)| = |\langle k_{\underline{\alpha}}(\cdot,x),g(\cdot)\rangle_{\underline{\alpha}}| \le ||k_{\underline{\alpha}}(\cdot,x)||_{\underline{\alpha}} \, ||g||_{\underline{\alpha}} =$$

$$\{(4\alpha_0\alpha_1)^{-\frac{1}{2}}(\alpha_0||g||_2^2 + \alpha_1||g'||_2^2)\}^{\frac{1}{2}} = \{(\tfrac{1}{2})(r||g||_2^2 + r^{-1}||g'||_2^2)\}^{\frac{1}{2}}$$

where $r^2 = {}^{\alpha}0/\alpha_1$, for all $\alpha_0,\alpha_1 > 0$. The right side is minimized by choosing $r = ||g'||_2/||g||_2$, yielding (2.2). $\square$

As a consequence of the necessary and sufficient condition for equality in the Cauchy-Schwarz inequality, equality in (2.2) is seen to hold if and only if g is proportional to $\exp\{-\lambda|x-a|\}$, $x,a \in \mathbb{R}$ and $\lambda > 0$.

As a corollary to Proposition 2.1 we derive an analogous result for functions defined on the half-life or on a finite interval. Denote by $||\cdot||_{2,I}$ and $||\cdot||_{\infty,I}$ the $L_2(I)$- and $\sup_I$-norms respectively where $I = (a,b) \in \mathbb{R}$.

Proposition 2.2. Let I be a possibly infinite interval (a,b), and let $g,g' \in L_2(I)$. Then

$$||g||_{\infty,I}^2 \le ||g||_{2,I} \, ||g'||_{2,I} + (g(a)^2 + g(b)^2) / 2 . \tag{2.3}$$

Proof: We first extend g to g^*_λ on $\mathbb{R}$, which under our assumptions is a member of $H^1(\mathbb{R})$, where

$$g^*_\lambda(x) \;\equiv\; g(a)g^{\ell}_{\lambda,a}(x) + g(x) + g(b)g^{u}_{\lambda,b}(x) \;, \quad \lambda > 0 \;,$$

with

$$g^{u}_{\lambda,a}(x) \;\equiv\; e^{-\lambda(x-a)} I_{[a,\infty)}(x) \quad \text{and} \quad g^{\ell}_{\lambda,a} \;\equiv\; e^{\lambda(x-a)} I_{(-\infty,a]}(x) \;,$$

$$\text{for } a \in \mathbb{R} \text{ and } \lambda > 0 \,.$$

Then from Proposition (2.1), we have

$$\begin{aligned} \|g\|^2_{\infty,I} &= \|g^*_\lambda\|^2_\infty \le \|g^*_\lambda\| \, \|g^{*\prime}_\lambda\|_2 \\ &= \{[\|g\|_2^2 + \tfrac{1}{\lambda}(g(a)^2 + g(b)^2)/2]\cdot[\|g'\|_2^2 + \lambda(g(a)^2 + g(b)^2)/2]\}^{1/2} \\ &= \{\|g\|_2^2\,\|g'\|_2^2 + ((g(a)^2 + g(b)^2)/2)(\|g\|_2^2\lambda + \|g'\|_2^2/\lambda) \\ &\qquad + ((g(a)^2 + g(b)^2)/2)^2\}^{1/2} \,, \end{aligned}$$

for all $\lambda > 0$. Hence,

$$\|g\|^2_\infty \le \min_{\lambda>0}\,(\|g^*_\lambda\|_2\,\|g^{*\prime}_\lambda\|_2) = \|g\|_2\,\|g'\|_2 + (g(a)^2 + g(b)^2)/2 \,,$$

since the minimum is achieved for $\lambda = \|g'\|_2/\|g\|_2$. □

Note that inequality (2.3) is satisfied with equality if g is exponential on the half-line or constant on a finite interval (a,b) . Also, if say $g(a) = 0$, then form (2.3) we have

$$\|g\|^2_\infty \le \|g\|_2\,\|g'\|_2 + g(b)^2/2 \le \|g\|_2\,\|g'\|_2 + \|g\|^2_\infty/2 \,.$$

which implies

$$(2.4) \qquad ||g||_\infty^2 \leq 2||g||_2 \, ||g'||_2 .$$

For functions $g \in H^1((0,+\infty))$ inequality (2.4) can also be seen to hold by applying inequality (2.2) to $g^*(x) \equiv g(x)I_{(a,\infty)} + g(2a-x)I_{(-\infty,a)}$, where I_A denotes the indicator function of the set A. It is then clear that, as a consequence of the necessary and suffieient condition for equality in the Cauchy-Schwarz inequality, equality in (2.4) with $g \in H^1((0,+\infty))$ holds if and only if g is exponential.

3. An application. Let f be a probability density function on a possibly infinite interval (a,b) and denote by $I(f) \equiv \int_a^b (f'(x))^2/f(x)\,dx$ its Fisher information. Then $I(f) = 4||g'||_{2,I}^2$, where $g = f^{\frac{1}{2}}$. As a corollary to Proposition (2.2) (which covers Proposition (2.1), we derive the following inequality involving Fisher information:

Corollary 3.1. Let $g = f^{\frac{1}{2}}$ be as in Proposition 2.2. Then

$$(3.1) \qquad \sup_{x \in (a,b)} f(x) \leq [I(f)^{\frac{1}{2}} + f(a) + f(b)]/2 .$$

Inequality (3.1) may be modified to allow for discontinuities of f within (a,b).

Remark 3.1. It might be of interest to note the following two characterizations of density functions with minimal Fisher information for a given maximum of the density function:

(i) For all f on $\mathbb{R}$ with $\sup_x f(x) = B$, $I(f) \geq 4B^2$ with "=" iff f is the density of a Laplace (doubly exponential) distribution (see the remarks following Proposition (2.1)).

(ii) For all f on R^+ with $\sup_x f(x) = B$, $I(f) \geq B^2$ with "=" if and only if f is exponential (see the remarks following inequality (2.4)).

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